Uniformization of varieties with log-canonical singularities

Benoît Cadorel - IECL, Nancy (France)

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The classical case

Theorem (Yau)

Let X be a complex projective manifold, with K_X ample. Then, we have the following Miyaoka-Yau inequality *:*

$$\Delta_{\mathrm{MY}}(X) := \big(2(n+1)c_2(T_X) - n\, c_1^2(T_X)\big) \cdot c_1(K_X)^{n-2} \geqslant 0.$$

In case of equality, $X = {}^{\Gamma}\big\backslash_{\mathbb{B}^n}$ is a ball quotient by a cocompact torsion free subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$).

Idea of proof

1) K_X ample $\Rightarrow \exists\, \omega_{KE} \in c_1(K_X)$, i.e. $-\mathrm{Ric}(h_{KE}) = \omega_{KE}$.

2) Write

$$\underbrace{\int_X ||i\Theta(h_{\mathrm{KE}})^{\perp_{\omega_{KE}}}||^2_{h_{\mathrm{KE}}}}_{\propto -\Delta_{\mathrm{MY}}(X)} \leqslant 0.$$

.

3) In case of equality, the vanishing of $i\Theta(h_{\mathrm{KE}})^{\perp_{\omega_{\mathrm{KE}}}}$ yields

$$h_{\mathrm{KE}} = h_{\mathbb{B}^n} + O(|z|^2) \quad \text{near any point of } X$$

4) Deduce that $\exists (\mathbb{B}^n, h_{\mathbb{B}^n}) \xrightarrow{\Psi} (X, h_{\mathrm{KE}})$, covering map (holomorphic and locally isometric everywhere).

The klt case

Theorem (Greb-Kebekus-Peternell-Taji, 2019 $\sim$ 2020)

Let X be a complex projective variety with klt singularities and K_X ample. Then one has

$$\Delta_{\mathrm{MY}}^{\mathrm{orb}}(X) := \big(2(n+1)c_{2,\mathrm{orb}}(T_X) - n\, c_{1,\mathrm{orb}}^2(T_X)\big) \cdot c_1(K_X)^{n-2} \geqslant 0.$$

In case of equality, $X \cong \Gamma \backslash \mathbb{B}^n$, where $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is a discrete cocompact subgroup without fixed point in codimension 1.

On klt and lc singularities

Let X be a germ of singularity with K_X $\mathbb{Q}$-Cartier. Let $\pi : \widehat{X} \to X$ be a resolution of singularities, and write

$$K_{\widehat{X}} \sim_{\mathbb{Q}} \pi^* K_X + \sum_E a_E E, \quad (a_E \in \mathbb{Q} : \text{ discrepencies}).$$

Then :

1. if all $a_E > -1$, the singularities are *log-terminal* (klt) ;
2. if all $a_E \geqslant -1$, the singularities are *log-canonical* (lc).

For example, if X is a cone over a smooth projective curve C, then one may have

$$g(X) = \underbrace{\underbrace{0}_{\text{klt}}, \quad 1,}_{\text{lc}} \qquad \underbrace{2, 3, \ldots}_{\text{more complicated singularities}} \quad .$$

klt singularities are quotient in codimension 2.

(quotient singularities : $\overset{\text{loc}}{\cong} G \backslash \mathbb{C}^n$, with G finite)

Definition of $\Delta^{\text{orb}}_{\text{MY}}(X)$ (for X klt with K_X ample)

There exists $U \subset X$ with $\operatorname{codim}(X - U) \geqslant 2$, such that U has a structure of orbifold.

Pick $S = H_1 \cap \ldots \cap H_{n-2}$ with generic $H_i \in |m_i K_X|$ $(m_i \gg 1)$.

Then $S \subset U$. In fact, S has a (smooth) stack structure, and $\mathcal{T}_X|_S$ is an orbifold vector bundle.

Let

$$\Delta^{\text{orb}}_{\text{MY}}(X) := \frac{1}{\prod_i m_i} \left(2(n+1)c_{2,\text{orb}}(T_X|_S) - n\, c^2_{1,\text{orb}}(T_X|_S)\right)$$

Other generalizations

- [Claudon-Guenancia-Graf, 22]
  Let (X, Δ) be a klt pair with $\Delta = \sum_i (1 - \frac{1}{m_i}) D_i$ and $K_X + \Delta$ ample.
  Then $\Delta_{\mathrm{MY}}^{\mathrm{orb}}(X, \Delta) \geqslant 0$, and if equality :
  $$X \cong \Gamma \backslash \mathbb{B}^n \quad (\Gamma \subset \mathrm{Aut}(\mathbb{B}^n) \text{ cocompact}).$$

- [Deng, 20]
  Let $X = \overline{X} - D$ be a quasi-projective variety with $\overline{X}, D$ smooth. Assume that $(\Omega_{\overline{X}}(\log D) \oplus \mathcal{O}_{\overline{X}}, \theta)$ is L-stable with respect to a nef and big L. Then we have a Miyaoka-Yau inequality w.r.t. L, and in case of equality
  $$X \cong \Gamma \backslash \mathbb{B}^n \quad (\Gamma \subset \mathrm{Aut}(\mathbb{B}^n) \text{ torsion free, not cocompact}).$$

Baily-Borel-Mok compactifications of non-compact ball quotient

Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a lattice (i.e. discrete, finite Bergman covolume).

Theorem (Baily-Borel (1966), Mok (2012))

The quotient $X := \Gamma \backslash \mathbb{B}^n$ *admits a structure of quasi-projective variety, and a projective compactification*

$$X^* \cong X \cup \{p_1, \ldots, p_m\}.$$

The singularities p_i *are lc, and* K_{X^*} *is ample.*

Up to replacing Γ *by* $\Gamma' \subset \Gamma$, *the analytic germs* $(X^* \ni p_i)$ *are locally isomorphic to*

$$\mathrm{Cone}(A_i, L_i) \qquad (A_i : \textit{ abelian variety}, L_i : \textit{ anti-ample on } A_i).$$

Can we characterize algebraically the X^* obtained this way ? By means of a Miyaoka-Yau inequality ?

(Note : the surface case is well-understood thanks to work of S. Kobayashi. We will focus on the case $n \geqslant 3$*)*

Setting

Let X^* be a projective variety with :

1. punctual, lc singularities (let $X := (X^*)^{\mathrm{reg}}$) ;
2. K_{X^*} ample ;

Miyaoka-Yau characteristic number

As before, pick $S = H_1 \cap \ldots \cap H_{n-2}$ with generic $H_i \in |m_i K_X|$ ($m_i \gg 1$).

Then $S \subset X$.

Let

$$\Delta_{\mathrm{MY}}(X^*) := \frac{1}{\prod_i m_i} \left(2(n+1)c_2(T_X|_S) - n\, c_1^2(T_X|_S)\right)$$

Theorem (Consequence of [Greb-Kebekus-Peternell-Taji])

In this situation, one has $\Delta_{\mathrm{MY}}(X^*) \geqslant 0$. In case of equality, there exists

$$\psi : \widetilde{X} \to \mathbb{B}^n,$$

étale everywhere, equivariant for a representation $\rho : \pi_1(X) \to \mathrm{Aut}(\mathbb{B}^n)$.

Idea of proof

1) [Guenancia, 15] Since K_{X*} is ample and X^* has lc singularities, $\mathcal{T}_{X*}$ is K_{X*}-semi-stable ;

2) [GKPT, 19] the Higgs sheaf $(\Omega^{[1]}_{X*} \oplus \mathcal{O}_{X*}, \theta)$ is K_{X*}-stable ;

3) For $S \subset X$ as above, the restriction $(\Omega^{[1]}_{X*} \oplus \mathcal{O}_{X*}, \theta)|_S$ is still K_{X*}-stable as a Higgs *bundle* (Mehta-Ramanathan type of argument) ;

4) [Simpson] One has $0 \leqslant \Delta_{\mathrm{BG}}(\Omega^{[1]}_{X*} \oplus \mathcal{O}_{X*}) = \Delta_{\mathrm{MY}}(X^*)$.
In case of equality, there exist

$$\begin{cases} \rho : \pi_1(S) \to \mathrm{Aut}(\mathbb{B}^n) \\ \psi_S : \widetilde{S} \to \mathbb{B}^n \quad \rho\text{-equivariant} \end{cases}$$

5) Sweep $\widetilde{X}$ by $\widetilde{S}$, and glue the different ψ_S to define

$$\psi : \widetilde{X} \to \mathbb{B}^n$$

(Idea of T. Mochizuki)

The proof still works if we replace "log-canonical singularities" by "$\mathcal{T}_{X*}$ is K_{X*}-semistable" (jump directly to Step 2).

Given $\psi : \widetilde{X} \to \mathbb{B}^n$ locally étale as above, we just need to chack that $\psi^* h_{\mathbb{B}^n}$ is complete to prove that ψ is an isomorphism.

This then implies that X^* is the Baily-Borel-Mok compactification of

$$\Gamma \Big\backslash \mathbb{B}^n \quad (\Gamma = \rho(\pi_1(X))).$$

A class of non-examples to avoid

Deligne, Mostow, Siu, Deraux, Stover, Toledo... have constructed compact Kähler manifolds M sitting in diagrams

$$\begin{tikzcd} & \widetilde{M} \arrow[r, "\psi"] \arrow[d, "\pi"] & \mathbb{B}^n \\ D_1, D_2, \ldots, D_m \arrow[r, hook] & M & \end{tikzcd}$$

where

1. the D_i are smooth disjoint divisors. Let $D = \sum_i D_i$;
2. $\psi|_{\pi^{-1}(M-D)}$ is ρ-equivariant for some $\rho : \pi_1(M - D) \to \mathrm{Aut}(\mathbb{B}^n)$, and is étale on this open subset of $\widetilde{M}$.
3. near each $\pi^{-1}(D_i)$, ψ has the form

$$\psi(z_1, z_2, \ldots, z_n) = (z_1^{m_i}, z_2, \ldots, z_n) \qquad (D_i \overset{loc}{=} \{z_1 = 0\}).$$

At least for Deraux's and Deligne-Mostow-Siu's examples :

Proposition (C, 24)

Let $X = M - D$. There exists a morphism $M \to X^*$, contracting each D_i to a point, and leaving X untouched. For X^*, we have the following :

1. *K_{X^*} is $\mathbb{Q}$-Cartier and ample ;*
2. *$\mathcal{T}_{X^*}$ is K_{X^*}-semistable ;*
3. *$\Delta_{\mathrm{MY}}(X^*) = 0$;*

However, $\psi : \widetilde{X} \to \mathbb{B}^n$ is not uniformizing.

Obtaining X^* as an abstract complex space is fairly easy. Showing that K_{X^*} is $\mathbb{Q}$-Cartier (and then ample) does not seem immediate.

However, the singularities of the X^* are not log-canonical here !

Statement of the criterion

Theorem (C, 24)

Let X^* be a complex projective variety of dimension $\geqslant 3$, with punctual log-canonical singularities,and K_{X^*} ample.
Assume :

1. $\Delta_{\mathrm{MY}}(X^*) = 0$ *;*
2. *there exists a log-resolution of singularities $Y \to X^*$ such that all $a_E = -1$.*

Then X^* is a Baily-Borel-Mok compactification of a ball quotient by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$.

As said above, the proof boils down to showing that $\psi^* h_{\mathbb{B}^n}$ is complete.

Asymptotic behaviour of the period map

Write $Y = X \cup D$, where D is an SNC divisor.

Let $F \subset D$ be a smooth (locally closed) stratum, and let $(\Delta^*)^k \times \Delta^{n-k} \subset X$ be an adapted polydisk so that $F \overset{loc}{\cong} \{0\}^k \times \Delta^{n-k}$.

We get a diagram

$$\begin{tikzcd} \mathbb{H}^k \times \Delta^{n-k} \arrow[r] \arrow[d, "\pi"] & \widetilde{X} \arrow[r, "\psi"] \arrow[d, "\pi"] & \mathbb{B}^n \\ (\Delta^*)^k \times \Delta^{n-k} \arrow[r, hook] & X & \end{tikzcd}$$

The *nilpotent orbit theorem* ([Schmid], [Cattani-Deligne-Kaplan] and recently [Sabbah-Schnell], [Deng]) shows that there are two possibilities :

1. $\exists b_\infty \in \partial\mathbb{B}^n$ such that
$$\psi(w) \underset{\pi(w)\to\{0\}^k\times\Delta^{n-k}}{\longrightarrow} b_\infty$$
2. $\exists \varphi : \{0\}^k \times \Delta^{n-k} \to \mathbb{B}^p \hookrightarrow \mathbb{B}^n$ (totally geodesic subball with $p < n$) such that
$$\psi(w) \underset{\pi(w)\to z\in\{0\}^k\times\Delta^{n-k}}{\longrightarrow} \varphi(z)$$

Above a point of a smooth stratum $F \subset D$:

1. either $\exists b_\infty \in \partial\mathbb{B}^n$ such that

$$\psi(w) \underset{\pi(w)\to\{0\}^k\times\Delta^{n-k}}{\longrightarrow} b_\infty$$

2. or $\exists\varphi : \{0\}^k \times \Delta^{n-k} \to \mathbb{B}^p \hookrightarrow \mathbb{B}^n$ (totally geodesic subball with $p < n$) such that

$$\psi(w) \underset{\pi(w)\to z\in\{0\}^k\times\Delta^{n-k}}{\longrightarrow} \varphi(z)$$

If Case 1 happens for all strata, then $\psi^* h_{\mathbb{B}^n}$ is complete.

Assume by contradiction that Case 2 happens for some F. Then glueing the local φ defined above, we may define

$$\begin{cases} \psi_F : \widetilde{F} \to \mathbb{B}^p \subset \mathbb{B}^n \\ \rho_F : \pi_1(F) \to \mathrm{Aut}(\mathbb{B}^p) \end{cases} \qquad \text{(equivariant data)}$$

We are going to show that these maps ψ_F are constant.

Varieties of special type according to Campana

"Definition"

Let $X = \overline{X} - D$ be a quasi-projective variety. We say that X is of *special type* if there exists no rational fibration $(\overline{X}, D) \overset{f}{\dashrightarrow} (Y, \Delta)$ of general type. (where (Y, Δ) is the *orbifold base* of f, according to Campana).

Theorem (Campana)

Let X be a quasi-projective variety. Then there exists a fibration $c : X \dashrightarrow (C, \Delta_c)$ *(the* core fibration*) where*

1. *the very general fibre of c is of* special type *;*
2. *the orbifold base (C, Δ_c) is of* general type*.*

Special connectedness and log-canonical singularities

Theorem (C, 24)

*Let X^* be a variety with log-canonical singularities, and let $q : Y \to X^*$ be a log-resolution. Then the fibers of q are connected by chains of varieties of special type (*specially chain connected*).*

[Hacon-McKernan, 05] For klt singularities, the fibers are *rationally chain connected*.

For the proof, substitute in [Hacon-McKernan] the use of the MRC fibration with the use of Campana's core fibration !

Corollary

Assume $a_E = -1$ for all q-exceptional divisors. Then the smooth strata F of the exceptional divisors are actually of special type.

Isotriviality of $\mathbb{C}$-VHS on varieties of special type

Theorem (Consequence of [C Deng Yamanoi 22])

Let F be a quasi-projective variety of special type. Then any polarized $\mathbb{C}$-VHS on F is isotrivial. In particular, if we are given

$$\begin{cases} \psi_F : \widetilde{F} \to \mathbb{B}^p \\ \rho_F : \pi_1(F) \to \mathrm{Aut}(\mathbb{B}^p) \end{cases} \qquad \textit{(equivariant data)},$$

then ψ_F is actually constant.

[Taji 13] Isotriviality for families of canonically polarized varieties above bases of special type.

ψ_F lands in a subdomain of $\mathbb{B}^p$, homogeneous under the semi-simple part of the (reductive) algebraic group $\overline{(\mathrm{Im}(\rho_F))}^{Zar}$.

But [C Deng Yamanoi] implies the latter is also virtually abelian, hence finite.

End of proof of the criterion

X^* has punctual singularities with all $a_E = -1$ for a log-resolution $q : Y \to X$.

Write $Y = X \cup D$.

For all stratum $F \subset D$ we have then *constant* limiting maps $\varphi : \widetilde{F} \to \overline{\mathbb{B}^n}$ (*cheating a bit : this map needs more data to be uniquely determined*).

We want to exclude the fact that such a φ_F lands in $\mathbb{B}^n \subset \overline{\mathbb{B}^n}$.

If this happens, the situation is actually as follows.

A contradiction

Let $p \in X^*$ lying under F and let $b_\infty \in \mathbb{B}^n$ be the limiting point. Then, we may find a radius $r > 0$ and a neighborhood $U \subset X$, such that

$$\begin{tikzcd}
& B(b_\infty, r) - \{b_\infty\} \arrow[dr, hook] & \\
\widetilde{U - \{p\}} \arrow[ur, "\text{covering}"] \arrow[r] \arrow[d] & \tilde{X} \arrow[r, "\psi"] \arrow[d] & \mathbb{B}^n \\
U - \{p\} \arrow[r, hook] & X &
\end{tikzcd}$$

commutes.

But $B(b_\infty, r) - \{b_\infty\}$ is simply connected ! So $\widetilde{U - \{p\}} \cong \mathbb{B}^n - \{0\}$, and

$$U \cong G \backslash \mathbb{B}^n \quad (G \text{ finite}).$$

The singularities are quotient, hence klt. Contradiction with the hypothesis $a_E = -1$.

Thank you for your attention !